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RESEARCH

New supersets of Zadoff-Chu sequences via the Weil bound

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Abstract
Sequences with low correlation have important applications in communications, radar, and cryptography. Recently, Pitaval et al. proposed two enlarged supersets of Zadoff-Chu (ZC) sequences with low correlation zone (LCZ) property to overcome physical random access channel (PRACH) capacity shortfall in 5G. In this paper, we present several classes of generalized supersets of ZC sequences using the Weil bound of exponential sums over finite fields. We employ ZC sequences and traditional sequences like m-sequences, binary Gold sequences and p-phase sequences as covers to propose several new classes of supersets of ZC sequences. The proposed sequence families have more flexible parameters compared to previous constructions. Interestingly, one of the resultant sequence sets is asymptotically optimal with respect to the Welch bound.

Keywords Binary Gold sequences · Exponential sums · Low correlation zone (LCZ) · m-sequences · p-phase sequences · Physical random access channel (PRACH) · Weil bound · Zadoff-Chu (ZC) sequences

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1 Introduction

The physical random access channel (PRACH) in a cellular network transmits uplink (UL) synchronisation signals to create initial link between a user equipment (UE) and a base station (BS). To identify several UEs accessing the network at the same time, multiple PRACH signals, also known as preambles, should be supported by the network. The preambles must have low out-of-phase auto-correlation and low cross-correlations in order for the BS to accurately recognise numerous preambles and estimate their timing for synchronisation.

In order to support new features such as high-speed transmissions, millimetre-wave, and low-latency, 5G NR introduces many new preamble formats, designed from a short sequence and wider subcarrier spacings (SCSs). This combination, however, dramatically decreases the number of PRACH preambles that can be used in a network [1, 2]. Working on this problem, the authors in [3] based on [4] redesigned the PRACH preambles from a totally new sequence set. Based on multi-preamble transmissions, several PRACH preamble designs were also proposed in [5–7]. However, none of the constructions improved the PRACH spectral efficiency [7].

To expand an existing sequence set which can be used as 5G PRACH preambles, the additional new sequences should be carefully chosen in order to maintain the correlation properties. Recently in [8, 9], Pitaval et al. designed the ZC_m and ZC_a sequence supersets with low correlation zone (LCZ) property to overcome PRACH capacity shortfall. The ZC_a and ZC_m sequence supersets are constructed by elementwise multiplication of Zadoff-Chu (ZC) sequences [10] with Alltop sequence covers and m-sequence covers of the same period, respectively [8, 9]. In both the constructions, correlation properties of the resultant sequences are similar to that of the ZC sequences, and set sizes are greatly increased compared to ZC sequences.

Motivated by the work of [8, 9], in search of large ZC supersets to overcome the PRACH capacity shortfall, we derive several new classes of generalized supersets of ZC sequences with low correlation and flexible parameters, from the Weil bound on exponential sums over finite fields. In 1948 Weil [11] proved the following celebrated fact.

Fact 1 ([11]) Let p be a prime number, m be a positive integer, and $\mathbb{F}_{p^m}$ denote the finite field with p^m elements. Let χ be a nontrivial additive character of $\mathbb{F}_{p^m}$ and $f(x) \in \mathbb{F}_{p^m}[x]$ with $\deg(f) = d \geq 1$ and $\gcd(d, p) = 1$. Then

$$\left| \sum_{x \in \mathbb{F}_{p^m}} \chi(f(x)) \right| \leq (d-1)\sqrt{p^m}. \tag{1}$$

Fact 2 ([12, 13]) Let χ be a nontrivial additive character of $\mathbb{F}_{p^m}$, and ψ be a nontrivial multiplicative character of $\mathbb{F}_{p^m}$ with order M and define $\psi(0) = 0$. Let $f(x) \in \mathbb{F}_{p^m}[x]$ with degree d, and $g(x) \in \mathbb{F}_{p^m}[x]$ with s distinct roots in $\overline{\mathbb{F}}_{p^m}$ where $g(x) \neq c \cdot h^M(x)$ for some $c \in \mathbb{F}_{p^m}$ and $h(x) \in \mathbb{F}_{p^m}[x]$. Then

$$\left| \sum_{x \in \mathbb{F}_{p^m}} \psi(g(x))\chi(f(x)) \right| \leq (d+s-1)\sqrt{p^m}. \tag{2}$$

In literature, the Weil bound [11] has been efficiently used to design several families of sequences of low correlaton [14–19]. Following those constructions, we construct four new classes of generalized supersets of ZC sequences with low correlation and flexible parameters. First, using the properties of trace function we propose two supersets of ZC sequences

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by elementwise multiplication of ZC sequences of period L with m-sequence covers of period N, where $N = p^m - 1 = LK$, p is a prime number, and m is a positive integer. The ZC_m sequence subset proposed in [8, 9] can be seen as a special case of one of the proposed constructions. Next, using the Weil bound of exponential sums, we propose two new classes of supersets of ZC sequences, ZC_{bG} and ZC_p. ZC_{bG} consists of ZC sequences with binary Gold sequence covers. ZC_p consists of ZC sequences with p-phase sequence covers. Table 1 gives an overview of the proposed and existing supersets of ZC sequences till date. From Table 1, it can be observed that the proposed sequence set in Theorem 1 provides a flexible tradeoff between the family size and alphabet size without changing the maximum correlation and the sequence period, and the proposed sequence sets in Theorem 2 and 3 have larger PRACH capacities than the previous ZC sequence set and its supersets.

The rest of this paper is organized as follows. In Section 2, we recall the basic notations and definitions. In Section 3, we propose ZC_m sequence sets which consist of ZC sequences of short period with m-sequence covers of long period, and give some examples. In Section 4, we propose ZC_{bG} sequence sets which consist of ZC sequences with binary Gold sequence covers, and give some examples. In Section 5, we propose ZC_p sequence sets which consist of ZC sequences with p-phase sequence covers, and give some examples. In Section 6, we compare the proposed supersets of ZC sequences with previous works. We conclude the paper in Section 7.

2 Preliminaries

Let us fix some notations before we begin.

- a^* denotes the conjugate of a complex number a;
- $\langle x \rangle_y$ denotes the least nonnegative residue of x modulo y for an integer x and a positive integer y;
- $\mathbb{F}_{p^m}$ denotes the finite field with p^m elements, where p is a prime number and m is a positive integer;
- $\zeta_n = e^{\frac{2\pi h\sqrt{-1}}{n}}$ denotes a primitive n-th complex root of unity, where $\gcd(h, n) = 1$;
- The period in this paper represents the minimum positive period.

Definition 1 Let $\mathbf{x} = (x(0), x(1), \cdots, x(N-1))$ and $\mathbf{y} = (y(0), y(1), \cdots, y(N-1))$ be two sequences of period N. Then the periodic crosscorrelation between $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$R_{\mathbf{x},\mathbf{y}}(\tau) = \sum_{n=0}^{N-1} x(n)y(\langle n+\tau \rangle_N)^*, \quad 0 \le |\tau| \le N-1. \tag{3}$$

When $\mathbf{x} = \mathbf{y}$, $R_{\mathbf{x},\mathbf{y}}(\tau)$ is called the autocorrelation of $\mathbf{x}$, and is denoted by $R_{\mathbf{x}}(\tau)$.

Definition 2 Let Z be a positive integer, and $\mathcal{S}$ be a family of M polyphase sequences of period N. The maximum correlation of $\mathcal{S}$ with a zone Z is defined by

$$\begin{aligned}\Theta(\mathcal{S}; Z) = \max\{|R_{\mathbf{x},\mathbf{y}}(\tau)| : \mathbf{x}, \mathbf{y} \in \mathcal{S}, (\mathbf{x} = \mathbf{y}, 0 < |\tau| < Z) \\ \text{or } (\mathbf{x} \ne \mathbf{y}, 0 \le |\tau| < Z)\}.\end{aligned} \tag{4}$$

Accordingly, $\mathcal{S}$ is said to be an $(N, M, Z, \Theta(\mathcal{S}; Z))$ sequence set.

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Table 1 Properties of proposed supersets compared to original ZC Set and ZC supersets of period N with LCZ of length N_{CS}

	# of sequences per subset	Intra-subset LCZ $\Theta(\mathcal{S}_k; N_{CS})$	# of subsets	Inter-subset Corr. $\Omega(\mathcal{S}_k, \mathcal{S}_j)$	PRACH capacity	Superset optimality factor $\rho^{(\mathcal{S})}$ when $N_{CS} = N$	Constraint
ZC[1]	$\lfloor \frac{N}{N_{CS}} \rfloor$	0	$N-1$	$\sqrt{N}$	$(N-1)\lfloor \frac{N}{N_{CS}} \rfloor$	1	N is a prime number.
	$(N-1)\lfloor \frac{N}{N_{CS}} \rfloor$	$\sqrt{N}$	1	n/a		1	
ZC_m[8]	$(N+1)\lfloor \frac{N}{N_{CS}} \rfloor$	$\sqrt{N+1}$	$N-1$	$\approx 2\sqrt{N+1} + \log_2(N+1)$	$(N^2-1)\lfloor \frac{N}{N_{CS}} \rfloor$	2	$N = p^m - 1$ is a prime number.
ZC_a[9]	$N\lfloor \frac{N}{N_{CS}} \rfloor$ one subset with $(N-1)\lfloor \frac{N}{N_{CS}} \rfloor$	$\sqrt{N}$	N	$\leq 2\sqrt{N}$	$(N^2-1)\lfloor \frac{N}{N_{CS}} \rfloor$	2	N is a prime number.
Theorem 1	$(N+1)\lfloor \frac{L}{N_{CS}} \rfloor$	$\sqrt{N+1}$	1	n/a	$(N+1)\lfloor \frac{L}{N_{CS}} \rfloor$	n/a	$N = p^m - 1 = LK$
Corollary 1	L	$\sqrt{N+1}$	1	n/a	L	1	$N = p^m - 1 = LK$
Theorem 2	$N\lfloor \frac{N}{N_{CS}} \rfloor$ one subset with $(N+1)\lfloor \frac{N}{N_{CS}} \rfloor$	$\leq 3\sqrt{N+1}$	$N+2$	$\leq 3\sqrt{N+1}$	$(N^2+2N+1)\lfloor \frac{N}{N_{CS}} \rfloor$	3	$N = 2^m - 1$, where m is an odd number.
Theorem 3	$N\lfloor \frac{N}{N_{CS}} \rfloor$ one subset with $(N+1)\lfloor \frac{N}{N_{CS}} \rfloor$	$\leq d\sqrt{N+1}$	$(N+1)^{d-1}$	$\leq d\sqrt{N+1}$	$\left(N(N+1)^{d-1}+1\right)\lfloor \frac{N}{N_{CS}} \rfloor$	d	$N = p^m - 1$, where p is an odd number.

1) p is a prime number and m is a positive integer;

2) d is a positive integer with $1 < d < p$; and

3) the superset optimality factor $\rho^{(\mathcal{S})}$ is defined in (15)

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Definition 3 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two disjoint sets, consisting of sequences of the same period N. Then the inter-set correlation is defined as

$$\Omega(\mathcal{S}_1, \mathcal{S}_2) = \max_{\substack{\mathbf{x}\in\mathcal{S}_1, \mathbf{y}\in\mathcal{S}_2 \\ 0\le|\tau|<N}} |R_{\mathbf{x},\mathbf{y}}(\tau)|. \tag{5}$$

Lemma 1 *The Welch bound* [20] *for* $\mathcal{S}$ *is given by*

$$\Theta(\mathcal{S}; N) \ge N\sqrt{\frac{M-1}{MN-1}}, \tag{6}$$

provided that $\sum_{n=0}^{N-1}|x(n)|^2 = N$ *(which is satisfied for polyphase sequences). We call the sequence set* $\mathcal{S}$ *optimal with respect to the Welch bound if* $\Theta(\mathcal{S}; N)$ *satisfies* (6) *with equality. For a given sequence set* $\mathcal{S}$*, we denote the Welch bound by* Θ_{opt}.

Definition 4 The well-known Zadoff-Chu (ZC) [10] sequences $\mathbf{x}_u = (x_u(0), x_u(1), \cdots, x_u(N-1))$ of period N with root u are defined as

$$x_u(n) = \zeta_N^{\frac{un(n+\langle N\rangle_2)}{2}+qn}, \ 0 \le n \le N-1, \tag{7}$$

where q and u are integers, with $\gcd(u, N) = 1$.

Definition 5 The absolute trace function Tr from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ is defined by

$$\mathrm{Tr}(x) = x + x^p + \cdots + x^{p^{m-1}}, \ x \in \mathbb{F}_{p^m}. \tag{8}$$

Definition 6 (Additive Character) An additive character of $\mathbb{F}_{p^m}$ is a nonzero function χ from $\mathbb{F}_{p^m}$ to the set of complex numbers such that $\chi(x+y) = \chi(x)\chi(y)$ for any pair $(x, y) \in \mathbb{F}_{p^m}^2$. For each $a \in \mathbb{F}_{p^m}$, the function

$$\chi(x) = \zeta_p^{\mathrm{Tr}(ax)}, \quad x \in \mathbb{F}_{p^m}, \tag{9}$$

defines an additive character of $\mathbb{F}_{p^m}$. When $a = 0$, $\chi(x) = 1$ for all $x \in \mathbb{F}_{p^m}$, and is called the trivial additive character of $\mathbb{F}_{p^m}$.

Definition 7 (Multiplicative Character) A multiplicative character of $\mathbb{F}_{p^m}$ is a nonzero function ψ from $\mathbb{F}_{p^m}^*$ to the set of complex numbers such that $\psi(xy) = \psi(x)\psi(y)$ for any pair $(x, y) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^m}^*$, where $\mathbb{F}_{p^m}^* = \mathbb{F}_{p^m}\backslash\{0\}$ denotes the multiplicative group of $\mathbb{F}_{p^m}$. For $M|(p^m-1)$ and each $j = 0, 1, \cdots, M-1$, the function ψ

$$\psi\left(\alpha^k\right) = \zeta_M^{jk}, \quad \text{for } k = 0, 1, \cdots, p^m - 2, \tag{10}$$

defines a multiplicative character of $\mathbb{F}_{p^m}$ of order M. When $j = 0$, $\psi(x) = 1$ for all $x \in \mathbb{F}_{p^m}^*$, and is called the trivial multiplicative character of $\mathbb{F}_{p^m}$.

Definition 8 Let p be a prime number, m be a positive integer and $N = p^m - 1$. Then the m-sequence [21] $\mathbf{c} = (c(0), c(1), \cdots, c(N-1))$ of period N as

$$c(n) = \chi(\alpha^n), \ 0 \le n \le N-1, \tag{11}$$

where χ denotes the additive canonical character of $\mathbb{F}_{p^m}$ and α is a primitive element of $\mathbb{F}_{p^m}$.

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Definition 9 Let m be an odd number and $N = 2^m - 1$. Then for $0 \le i \le 2^m$, the Gold [22] sequences $\mathbf{a}_i = (a_i(0), a_i(1), \cdots, a_i(N-1))$ of period N can be defined as

$$a_i(n) = \begin{cases} \chi(\alpha^{i+n} + \alpha^{3n}), & \text{for } 0 \le i \le 2^m - 2, \\ \chi(\alpha^{3n}), & \text{for } i = 2^m - 1, \\ \chi(\alpha^n), & \text{for } i = 2^m, \end{cases} \tag{12}$$

where χ denotes the additive canonical character of $\mathbb{F}_{2^m}$ and α is a primitive element of $\mathbb{F}_{2^m}$.

Definition 10 Let p be an odd prime number, m and d be two positive integers with $1 < d < p$, and $N = p^m - 1$. For all $\Delta = (\delta_1, \delta_2, \cdots, \delta_{d-1}) \in \mathbb{F}_{p^m}^{d-1}$, the sequences $\mathbf{b}_\Delta = (b_\Delta(0), b_\Delta(1), \cdots, b_\Delta(N-1))$ of period N can be defined as

$$b_\Delta(n) = \chi(\alpha^n + \delta_1\alpha^{2n} + \delta_2\alpha^{3n} + \cdots + \delta_{d-1}\alpha^{dn}), \quad 0 \le n \le N-1, \tag{13}$$

where χ denotes the additive canonical character of $\mathbb{F}_{p^m}$ and α is a primitive element of $\mathbb{F}_{p^m}$.

Lemma 2 ([12]) *Let p be a prime number, m be a positive integer, and $a \in \mathbb{F}_{p^m}$. Then*

$$\sum_{n=0}^{p^m-2} \chi(a\alpha^n) = \begin{cases} p^m - 1, & a = 0, \\ -1, & a \ne 0. \end{cases} \tag{14}$$

where χ denotes the additive canonical character of $\mathbb{F}_{p^m}$ and α is a primitive element of $\mathbb{F}_{p^m}$.

Definition 11 For a given superset $\mathcal{S}$ of ZC sequences, consisting of sequences of period N, the superset optimality factor $\rho^{(\mathcal{S})}$ is defined as

$$\rho^{(\mathcal{S})} = \lim_{N\to\infty} \frac{\Theta(\mathcal{S}; N)}{\Theta_{opt}}. \tag{15}$$

When $\rho^{(\mathcal{S})} = 1$, we say that $\mathcal{S}$ is asymptotically optimal with respect to the Welch bound.

3 ZC Sequences of short period with *m*-sequence covers (ZC$_m$)

In this section, we use m-sequence to cover ZC sequences of short period and propose two families of sequences with low correlation. We hereby define $N = p^m - 1 = LK$ until the end of this section to avoid definition confusion, where p is a prime number and m, L, K are three positive integers.

According to (7), the ZC sequence $\mathbf{y}_u = (y_u(0), y_u(1), \cdots, y_u(L-1))$ of period L is defined as

$$y_u(n) = \zeta_L^{\frac{un(n+\langle L\rangle_2)}{2}+qn}, \quad 0 \le n \le L-1. \tag{16}$$

$\mathbf{y}_u$ is shifted with a constant value $0 < N_{CS} \le L$ such that it enables unambiguous detection with timing uncertainty $|\tau| < N_{CS}$. Then cyclically-shifted ZC sequences are generated as

$$y_{(u,v)}(n) = y_u(\langle n + C_v\rangle_L), \quad 0 \le n \le L-1, \tag{17}$$

where $C_v = vN_{CS}$ with $0 \le v \le \lfloor \frac{L}{N_{CS}} \rfloor - 1$.

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Next, we construct new sequences $\mathbf{z}_{(u,v,w)}$ of period N by elementwise multiplication of $\mathbf{y}_{(u,v)}$ with $\mathbf{c}_w$ for $0 \le w \le N-1$, as

$$z_{(u,v,w)}(n) = y_{(u,v)}(n) \cdot c_w(n), \quad 0 \le n \le N-1, \tag{18}$$

where

$$c_w(n) = c(\langle n+w \rangle_N), \quad 0 \le n \le N-1, \tag{19}$$

are cyclic-shifts of m-sequences, given in (11) and u is a constant with $gcd(u, L) = 1$.

Theorem 1 *Let p be a prime number, and N, m, L, K be four positive integers satisfying $N = p^m - 1 = LK$. Let N_{CS} be an integer with $0 < N_{CS} \le L$ and u be a constant with $gcd(u, L) = 1$. Then, the sequence family*

$$\mathcal{S}_1 = \{\mathbf{y}_{(u,v)}\}_{v=0}^{\lfloor \frac{L}{N_{CS}} \rfloor - 1} \bigcup_{w=0}^{N-1} \{\mathbf{z}_{(u,v,w)}\}_{v=0}^{\lfloor \frac{L}{N_{CS}} \rfloor - 1},$$

is an $\left(N, (N+1)\lfloor \frac{L}{N_{CS}} \rfloor, N_{CS}, \sqrt{N+1}\right)$ sequence set with the following properties:

- *If $\mathbf{y} = \mathbf{y}_{(u,v)}$ and $\mathbf{y}' = \mathbf{y}_{(u,v')}$, then we have*

$$|R_{\mathbf{y},\mathbf{y}'}(\tau)| = \begin{cases} N, & \text{if } L \mid C_v - C_{v'} - \tau, \\ 0, & \text{if } L \nmid C_v - C_{v'} - \tau. \end{cases} \tag{20}$$

- *If $\mathbf{y} = \mathbf{y}_{(u,v)}$ and $\mathbf{z} = \mathbf{z}_{(u,v',w)}$, then we have*

$$|R_{\mathbf{y},\mathbf{z}}(\tau)| = \begin{cases} 1, & \text{if } L \mid C_v - C_{v'} - \tau, \\ \sqrt{N+1}, & \text{if } L \nmid C_v - C_{v'} - \tau, \end{cases} \tag{21}$$

and

$$|R_{\mathbf{z},\mathbf{y}}(\tau)| = \begin{cases} 1, & \text{if } L \mid C_{v'} - C_v - \tau, \\ \sqrt{N+1}, & \text{if } L \nmid C_{v'} - C_v - \tau. \end{cases} \tag{22}$$

- *If $\mathbf{x} = \mathbf{z}_{(u,v,w)}$ and $\mathbf{x}' = \mathbf{z}_{(u,v',w')}$, then we have*

$$|R_{\mathbf{x},\mathbf{x}'}(\tau)| = \begin{cases} N, & \text{if } \tau = \langle w - w' \rangle_N \text{ and } L \mid C_v - C_{v'} - \tau, \\ 0, & \text{if } \tau = \langle w - w' \rangle_N \text{ and } L \nmid C_v - C_{v'} - \tau, \\ 1, & \text{if } \tau \ne \langle w - w' \rangle_N \text{ and } L \mid C_v - C_{v'} - \tau, \\ \sqrt{N+1}, & \text{if } \tau \ne \langle w - w' \rangle_N \text{ and } L \nmid C_v - C_{v'} - \tau, \end{cases} \tag{23}$$

where $\mathbf{y}_{(u,v)}$ and $\mathbf{z}_{(u,v,w)}$ are given in (17) and (18) respectively.

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Proof For $\mathbf{y} = \mathbf{y}_{(u,v)}$ and $\mathbf{y}' = \mathbf{y}_{(u,v')}$, we have

$$\begin{aligned} |R_{\mathbf{y},\mathbf{y}'}(\tau)| &= |\zeta_L^{\frac{u(C_v - C_{v'} - \tau)(C_v + C_{v'} + \tau + \langle L \rangle_2)}{2} + q(C_v - C_{v'} - \tau)} \cdot \sum_{n=0}^{N-1} \zeta_L^{un(C_v - C_{v'} - \tau)}| \\ &= |\sum_{n=0}^{N-1} \zeta_L^{un(C_v - C_{v'} - \tau)}| \\ &= \begin{cases} N, & \text{if } L \mid C_v - C_{v'} - \tau, \\ 0, & \text{if } L \nmid C_v - C_{v'} - \tau. \end{cases} \end{aligned} \tag{24}$$

Hence (20) is derived.

For $\mathbf{y} = \mathbf{y}_{(u,v)}$ and $\mathbf{z} = \mathbf{z}_{(u,v',w)}$, we have

$$\begin{aligned} |R_{\mathbf{y},\mathbf{z}}(\tau)| &= \left| \zeta_L^{\frac{u(C_v - C_{v'} - \tau)(C_v + C_{v'} + \tau + \langle L \rangle_2)}{2} + q(C_v - C_{v'} - \tau)} \right. \\ &\quad \left. \cdot \sum_{n=0}^{N-1} \zeta_L^{un(C_v - C_{v'} - \tau)} \cdot \chi(-\alpha^{n+\tau+w}) \right| \\ &= \left| \sum_{n=0}^{N-1} \zeta_L^{un(C_v - C_{v'} - \tau)} \cdot \chi(-\alpha^{n+\tau+w}) \right|. \end{aligned} \tag{25}$$

When $L \mid C_v - C_{v'} - \tau$, using Lemma 2, we get from (25),

$$\begin{aligned} |R_{\mathbf{y},\mathbf{z}}(\tau)| &= \left| \sum_{n=0}^{N-1} \chi(-\alpha^{n+\tau+w}) \right| = \left| \sum_{n=0}^{p^m-2} \chi(-\alpha^{\tau+w} \cdot \alpha^n) \right| \\ &= 1. \end{aligned} \tag{26}$$

When $L \nmid C_v - C_{v'} - \tau$, squaring both sides of (25), we have

$$\begin{aligned} |R_{\mathbf{y},\mathbf{z}}(\tau)|^2 &= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \chi(\alpha^{\tau+w}(\alpha^{n_2} - \alpha^{n_1})) \cdot \zeta_L^{u(n_1-n_2)(C_v - C_{v'} - \tau)} \\ &= \sum_{n_1=0}^{N-1} \sum_{j=1-N+n_1}^{n_1} \chi(\alpha^{\tau+w}(\alpha^{n_1-j} - \alpha^{n_1})) \cdot \zeta_L^{uj(C_v - C_{v'} - \tau)} \\ &= \sum_{n_1=0}^{N-1} \sum_{j=0}^{N-1} \chi(\alpha^{\tau+w}(\alpha^{n_1-j} - \alpha^{n_1})) \cdot \zeta_L^{uj(C_v - C_{v'} - \tau)} \\ &= \sum_{j=0}^{N-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \sum_{n_1=0}^{p^m-2} \chi(\alpha^{\tau+w}(\alpha^{n_1-j} - \alpha^{n_1})) \\ &= p^m - 1 + \sum_{j=1}^{N-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \cdot (-1) \text{ (using Lemma 2)} \\ &= p^m + \sum_{j=0}^{N-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \cdot (-1) \\ &= p^m = N + 1. \end{aligned} \tag{27}$$

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Therefore we have $|R_{\mathbf{y},\mathbf{z}}(\tau)| = \sqrt{N+1}$, and hence, (21) is obtained. The results in (22) can be obtained similarly.

Next, consider $\mathbf{x} = \mathbf{z}_{(u,v,w)}$ and $\mathbf{x}' = \mathbf{z}_{(u,v',w')}$, then we have

$$\begin{aligned} |R_{\mathbf{x},\mathbf{x}'}(\tau)| &= \Bigg| \zeta_L^{\frac{u(C_v - C_{v'} - \tau)(C_v + C_{v'} + \tau + \langle L \rangle_2)}{2} + q(C_v - C_{v'} - \tau)} \\ &\quad \cdot \sum_{n=0}^{N-1} \zeta_L^{un(C_v - C_{v'} - \tau)} \cdot \chi(\alpha^n(\alpha^w - \alpha^{w'+\tau})) \Bigg| \\ &= \Bigg| \sum_{n=0}^{N-1} \zeta_L^{un(C_v - C_{v'} - \tau)} \cdot \chi(\alpha^n(\alpha^w - \alpha^{w'+\tau})) \Bigg|. \end{aligned} \tag{28}$$

For $\tau = \langle w - w' \rangle_N$ and $L \mid C_v - C_{v'} - \tau$, we have $\zeta_L^{un(C_v - C_{v'} - \tau)} = 1$ and $\chi(\alpha^n(\alpha^w - \alpha^{w'+\tau})) = 1$ for any $0 \le n \le N-1$. Hence from (28) we have $|R_{\mathbf{x},\mathbf{x}'}(\tau)| = N$. For $\tau = \langle w - w' \rangle_N$ and $L \nmid C_v - C_{v'} - \tau$, we have $\chi(\alpha^n(\alpha^w - \alpha^{w'+\tau})) = 1$ for any $0 \le n \le N-1$. Hence from (28) we have $|R_{\mathbf{x},\mathbf{x}'}(\tau)| = 0$. For $\tau \ne \langle w - w' \rangle_N$ and $L \mid C_v - C_{v'} - \tau$, we have $\zeta_L^{un(C_v - C_{v'} - \tau)} = 1$ for any $0 \le n \le N-1$. Hence from (28) we have $|R_{\mathbf{x},\mathbf{x}'}(\tau)| = |\sum_{n=0}^{N-1} \chi(\alpha^n(\alpha^w - \alpha^{w'+\tau}))| = 1$.

For $\tau \ne \langle w - w' \rangle_N$ and $L \nmid C_v - C_{v'} - \tau$. Squaring both sides of (28), we have

$$\begin{aligned} &|R_{\mathbf{x},\mathbf{x}'}(\tau)|^2 \\ &= \sum_{n_2=0}^{N-1} \sum_{n_1=0}^{N-1} \zeta_L^{u(n_1-n_2)(C_v - C_{v'} - \tau)} \cdot \chi((\alpha^{n_1} - \alpha^{n_2})(\alpha^w - \alpha^{w'+\tau})) \\ &= \sum_{n_2=0}^{N-1} \sum_{j=-n_2}^{N-n_2-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \cdot \chi((\alpha^{n_2+j} - \alpha^{n_2})(\alpha^w - \alpha^{w'+\tau})) \\ &= \sum_{n_2=0}^{N-1} \sum_{j=0}^{N-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \cdot \chi((\alpha^{n_2+j} - \alpha^{n_2})(\alpha^w - \alpha^{w'+\tau})) \\ &= \sum_{j=0}^{N-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \sum_{n_2=0}^{p^m-2} \chi((\alpha^{n_2+j} - \alpha^{n_2})(\alpha^w - \alpha^{w'+\tau})) \\ &= p^m - 1 + \sum_{j=1}^{N-1} \zeta_L^{uj(C_v - C_{v'} - \tau)} \cdot (-1) \text{ (using Lemma 2)} \\ &= p^m = N + 1. \end{aligned} \tag{29}$$

Therefore, we have $|R_{\mathbf{x},\mathbf{x}'}(\tau)| = \sqrt{N+1}$, and hence (23) is obtained. □

Example 1 Let $p = 3$, $m = 6$, $q = 1$, $N_{CS} = 60$, and then let $N = 3^6 - 1 = 182 \times 4$, i.e. $L = 182$ and $K = 4$. Various periodic correlations between the sequences of $\mathcal{S}_1$ are shown in Fig. 1. It displays the auto-correlation of $\mathbf{z}_{(1,1,1)}$, the cross-correlation between $\mathbf{y}_{(1,0)}$ and $\mathbf{z}_{(1,1,20)}$, and the cross-correlation between $\mathbf{y}_{(1,0)}$ and $\mathbf{y}_{(1,2)}$.

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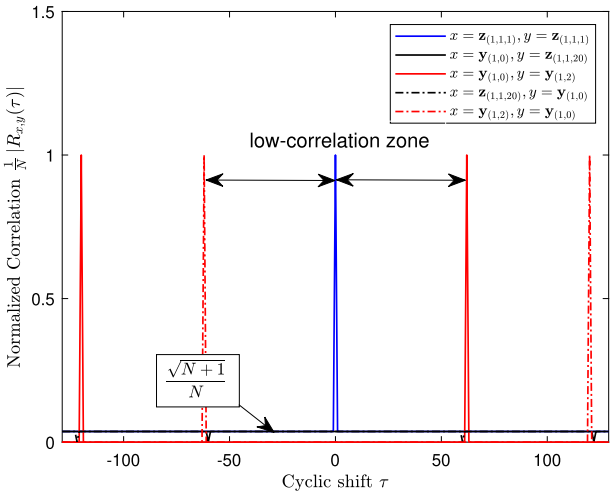

Fig. 1 Correlations between sequences of $\mathcal{S}_1$, $N = 728$

Corollary 1 *The sequence family* $\mathcal{S}_2 = \left\{\mathbf{z}_{(u,0,w)}\right\}_{w=0}^{L-1} \subset \mathcal{S}_1$ *is a sequence set with parameters* $(N, L, N, \sqrt{N+1})$*, and it has the following properties:*

$$|R_{\mathbf{z},\mathbf{z}'}(\tau)| = \begin{cases} N, & \text{if } \tau = \langle w - w' \rangle_N \text{ and } L \mid \tau, \\ 0, & \text{if } \tau = \langle w - w' \rangle_N \text{ and } L \nmid \tau, \\ 1, & \text{if } \tau \neq \langle w - w' \rangle_N \text{ and } L \mid \tau, \\ \sqrt{N+1}, & \text{if } \tau \neq \langle w - w' \rangle_N \text{ and } L \nmid \tau, \end{cases} \tag{30}$$

where $\mathbf{z} = \mathbf{z}_{(u,0,w)}$ *and* $\mathbf{z}' = \mathbf{z}_{(u,0,w')}$.

Proof The proof is similar to Theorem 1. Hence omitted. □

3.1 Discussion on optimality

For the sequence family $\mathcal{S}_2$ in Corollary 1, consider K as a constant integer, thus L increases as N increases. Let Θ_{opt} denote the optimal correlation value with respect to the Welch bound. For a fixed K, when $N \to \infty$, we have the optimality ratio, as follows

$$\begin{aligned} \rho^{\mathcal{S}_2} = \lim_{N\to\infty} \frac{\Theta(\mathcal{S}_2;N)}{\Theta_{opt}} &= \lim_{N\to\infty} \frac{\sqrt{N+1}}{\sqrt{\frac{N^2(L-1)}{LN-1}}} \\ &= \lim_{N\to\infty} \sqrt{\frac{N^3+N^2-KN-K}{N^3-KN^2}} \\ &= 1, \end{aligned} \tag{31}$$

which means that $\mathcal{S}_2$ is asymptotically optimal with respect to the Welch bound. This can also be intuitively verified by Table 2.

Example 2 Take $p = 2$, $m = 9$, $q = 1$, $L = 73$, $K = 7$, and $N = 2^9 - 1 = 73 \times 7$. Various periodic correlations between the sequences of $\mathcal{S}_2$ are shown in Fig. 2. It displays the auto-correlation of $\mathbf{z}_{(1,0,0)}$ and the cross-correlation between $\mathbf{z}_{(1,0,20)}$ and $\mathbf{z}_{(1,0,72)}$.

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Table 2 For $K = 4$, the comparison between $\Theta(\mathcal{S}_2; N)$ and Θ_{opt}

N	L	$\Theta(\mathcal{S}_2; N)$	Θ_{opt}
$3^2 - 1$	2	3	2.0656
$3^6 - 1$	182	27	26.9073
$3^{10} - 1$	14762	243	242.9897
$3^{12} - 1$	132860	729	728.9966

4 ZC Sequences with binary Gold sequence covers (ZC_{bG})

In this section, we use binary Gold sequences to cover ZC sequences and propose a family of sequences with LCZ. We hereby define $N = 2^m - 1$ until the end of this section to avoid definition confusion, where m is an odd number.

Cyclically-shifted ZC sequences $\mathbf{x}_{(u,v)}$ of period N are defined as

$$x_{(u,v)}(n) = x_u(\langle n + C_v\rangle_N) = \zeta_N^{\frac{u\langle n+C_v\rangle_N(\langle n+C_v\rangle_N+\langle N\rangle_2)}{2}+q\langle n+C_v\rangle_N}, \quad 0 \le n \le N-1, \quad (32)$$

where $C_v = vN_{CS}$ with $0 < N_{CS} \le N$ and $0 \le v \le \left\lfloor \frac{N}{N_{CS}} \right\rfloor - 1$.

Then the ZC_{bG} sequences $\mathbf{d}_{(u,v,i,w)} = (d_{(u,v,i,w)}(0), d_{(u,v,i,w)}(1), \cdots, d_{(u,v,i,w)}(N-1))$ are constructed by element-by-element multiplication of sequences $\mathbf{x}_{(u,v)}$ with covers as

$$d_{(u,v,i,w)}(n) = x_{(u,v)}(n) \cdot a_{(i,w)}(n), \quad 0 \le n \le N-1, \quad (33)$$

where the cover sequences $\mathbf{a}_{(i,w)} = (a_{(i,w)}(0), a_{(i,w)}(1), \cdots, a_{(i,w)}(N-1))$ with $0 \le i \le 2^m$ and $0 \le w \le N-1$ are cyclic-shifts of the cover in (12)

$$a_{(i,w)}(n) = a_i(\langle n + w\rangle_N), \quad 0 \le n \le N-1. \quad (34)$$

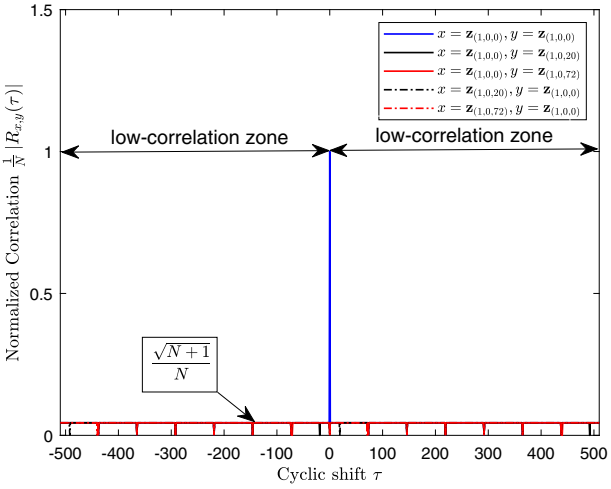

Fig. 2 Correlations between sequences of $\mathcal{S}_2$, $N = 511$

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From this definition, we generate the following set

$$\mathcal{D} = \bigcup_{i=0}^{2^m} \mathcal{D}_i, \tag{35}$$

where the last subset with $i = 2^m$, which contains cyclically-shifted ZC sequences and ZC sequences with m-sequence covers, is defined as

$$\mathcal{D}_{2^m} = \{\mathbf{x}_{(u,v)}\}_{v=0}^{\lfloor \frac{N}{N_{CS}} \rfloor - 1} \bigcup_{w=0}^{N-1} \{\mathbf{d}_{(u,v,2^m,w)}\}_{v=0}^{\lfloor \frac{N}{N_{CS}} \rfloor - 1}, \tag{36}$$

and for $0 \leq i < 2^m$, subsets are defined as

$$\mathcal{D}_i = \bigcup_{w=0}^{N-1} \{\mathbf{d}_{(u,v,i,w)}\}_{v=0}^{\lfloor \frac{N}{N_{CS}} \rfloor - 1}, \tag{37}$$

where u is a constant with $gcd(u, N) = 1$ for all i.

In the above construction, we use the parameter i of the Gold sequences as the index of sequence subsets, and generate $N + 2$ sequence subsets. In the sequence subset with index $i = 2^m$, not only ZC_{bG} sequences, but also cyclically-shifted ZC sequences are included. Then for sequence subsets with index $0 \leq i \leq 2^m - 1$, the size of their sequence sets is $N\lfloor \frac{N}{N_{CS}} \rfloor$, and for the sequence subset with index 2^m, the size of sequence set is $(N + 1)\lfloor \frac{N}{N_{CS}} \rfloor$. We refer to the PRACH capacity as the total number of sequences available for ZC_{bG} sequences, so we can get

$$\mathrm{PRACHcapacity}^{(\mathrm{ZC}_{bG})} = (N^2 + 2N + 1)\lfloor \frac{N}{N_{CS}} \rfloor. \tag{38}$$

At the same time, the collection of $N + 2$ subsets has the following properties:

Theorem 2 *Let m be an odd number and $N = 2^m - 1$. Let N_{CS} be an integer with $0 < N_{CS} \leq N$, for each subset $\mathcal{D}_i$ in (36) and (37) we have*

$$\Theta(\mathcal{D}_i; N_{CS}) \leq 3\sqrt{N+1}, \tag{39}$$

and for any two subsets $\mathcal{D}_i$ and $\mathcal{D}_j$, $i \neq j$, we have

$$\Omega\left(\mathcal{D}_i, \mathcal{D}_j\right) \leq 3\sqrt{N+1}. \tag{40}$$

Proof Let us consider the correlation between $\mathcal{D}_{i_1}$ and $\mathcal{D}_{i_2}$. For $0 \leq i_1, i_2 \leq 2^m - 2$, we have

$$\begin{aligned}
&|R_{\mathbf{d}_{(u,v_1,i_1,w_1)},\mathbf{d}_{(u,v_2,i_2,w_2)}}(\tau)| \\
&= \left| \sum_{n=0}^{N-1} \zeta_N^{un(C_{v_1} - C_{v_2} - \tau)} \cdot \chi\left(\alpha^{i_1+n+w_1} + \alpha^{3(n+w_1)} - \alpha^{i_2+n+w_2+\tau} - \alpha^{3(n+w_2+\tau)}\right) \right| \\
&= \left| \sum_{n=0}^{N-1} \zeta_N^{un(C_{v_1} - C_{v_2} - \tau)} \cdot \chi\left(\alpha^n(\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau}) + \alpha^{3n}(\alpha^{3w_1} - \alpha^{3(w_2+\tau)})\right) \right|. \qquad (41)
\end{aligned}$$

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When $\langle C_{v_1} - C_{v_2} - \tau\rangle_N = 0$ and $\langle w_1 - w_2 - \tau\rangle_N = 0$, we have $\alpha^{3w_1} - \alpha^{3(w_2+\tau)} = 0$ and $\zeta_N^{un(C_{v_1}-C_{v_2}-\tau)} = 1$. Using Lemma 2, the left side of (41) is equal to

$$\begin{aligned} |R_{\mathbf{d}_{(u,v_1,i_1,w_1)},\mathbf{d}_{(u,v_2,i_2,w_2)}}(\tau)| &= \Big| \sum_{n=0}^{N-1} \chi\left(\alpha^n(\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})\right) \Big| \\ &= \begin{cases} N, & \text{if} \quad i_1 = i_2, \\ 1, & \text{if} \quad i_1 \neq i_2. \end{cases} \end{aligned} \tag{42}$$

When $\langle C_{v_1} - C_{v_2} - \tau\rangle_N = 0$ and $\langle w_1 - w_2 - \tau\rangle_N \neq 0$, we know that $\zeta_N^{un(C_{v_1}-C_{v_2}-\tau)} = 1$ and $\alpha^{w_1} \neq \alpha^{w_2+\tau}$. Since $(3, 2^m - 1) = 1$, we can also get $\alpha^{3w_1} \neq \alpha^{3(w_2+\tau)}$. So (41) can be written as

$$\begin{aligned} &|R_{\mathbf{d}_{(u,v_1,i_1,w_1)},\mathbf{d}_{(u,v_2,i_2,w_2)}}(\tau)| \\ &= \Big| \sum_{n=0}^{N-1} \chi\left(\alpha^n(\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau}) + \alpha^{3n}(\alpha^{3w_1} - \alpha^{3(w_2+\tau)})\right) \Big| \\ &= \Big| \sum_{x\in\mathbb{F}_{2^m}^*} \chi\left((\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})x + (\alpha^{3w_1} - \alpha^{3(w_2+\tau)})x^3\right) \Big| \\ &= \Big| -1 + \sum_{x\in\mathbb{F}_{2^m}} \chi\left((\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})x + (\alpha^{3w_1} - \alpha^{3(w_2+\tau)})x^3\right) \Big| \\ &\leq 2\sqrt{2^m} + 1 \text{ (using Fact 1)} \\ &= 2\sqrt{N+1} + 1. \end{aligned} \tag{43}$$

When $\langle C_{v_1} - C_{v_2} - \tau\rangle_N \neq 0$ and $\langle w_1 - w_2 - \tau\rangle_N = 0$, then $\alpha^{3w_1} - \alpha^{3(w_2+\tau)} = 0$. Let $s = (\langle u(C_{v_1} - C_{v_2} - \tau)\rangle_N, N)$, $M = N/s$, $r = \langle u(C_{v_1} - C_{v_2} - \tau)\rangle_N/s$. Then (41) can be written as

$$\begin{aligned} |R_{\mathbf{d}_{(u,v_1,i_1,w_1)},\mathbf{d}_{(u,v_2,i_2,w_2)}}(\tau)| &= \Big| \sum_{n=0}^{N-1} \zeta_M^{rn} \cdot \chi\left(\alpha^n(\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})\right) \Big| \\ &= \Big| \sum_{x\in\mathbb{F}_{2^m}} \psi(x) \cdot \chi\left((\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})x\right) \Big| \\ &\begin{cases} = \left| s\sum_{n=0}^{M-1} \zeta_M^{rn} \right| = 0, & \text{if} \quad i_1 = i_2, \\ \leq \sqrt{2^m} = \sqrt{N+1}, & \text{if} \quad i_1 \neq i_2 \text{ (using Fact 2)}, \end{cases} \end{aligned} \tag{44}$$

where the inequality comes from the fact that $g(x) = x$ with one root in $\overline{\mathbb{F}}_{2^m}$ and $M \neq 1$, at the same time, $f(x) = (\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})x$ with degree $d = 1$.

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When $\langle C_{v_1} - C_{v_2} - \tau\rangle_N \neq 0$ and $\langle w_1 - w_2 - \tau\rangle_N \neq 0$, then (41) can be written as

$$\begin{aligned}
&|R_{\mathbf{d}_{(u,v_1,i_1,w_1)},\mathbf{d}_{(u,v_2,i_2,w_2)}}(\tau)| \\
&= \Big|\sum_{n=0}^{N-1} \zeta_M^{rn} \cdot \chi\left((\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})\alpha^n + (\alpha^{3w_1} - \alpha^{3(w_2+\tau)})\alpha^{3n}\right)\Big| \\
&= \Big|\sum_{x\in\mathbb{F}_{2^m}} \psi(x)\chi\left((\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})x + (\alpha^{3w_1} - \alpha^{3(w_2+\tau)})x^3\right)\Big| \\
&\leq 3\sqrt{2^m} \text{ (using Fact 2)} \\
&= 3\sqrt{N+1}, \qquad (45)
\end{aligned}$$

where the inequality comes from the fact that $g(x) = x$ with one root in $\overline{\mathbb{F}}_{2^m}$ and $M \neq 1$, at the same time, $f(x) = (\alpha^{i_1+w_1} - \alpha^{i_2+w_2+\tau})x + (\alpha^{3w_1} - \alpha^{3(w_2+\tau)})x^3$ with degree $d = 3$.

Therefore, we get that $\Theta(\mathcal{D}_i; N_{CS}) \leq 3\sqrt{N+1}$ holds for $0 \leq i \leq 2^m - 2$ from (44) and (45). At the same time, we can also get that $\Omega\left(\mathcal{D}_i, \mathcal{D}_j\right) \leq 3\sqrt{N+1}$ holds for $0 \leq i \neq j \leq 2^m - 2$ from above four cases. Similarly, we can also get that $\Theta(\mathcal{D}_i; N_{CS}) \leq 3\sqrt{N+1}$ and $\Omega\left(\mathcal{D}_i, \mathcal{D}_j\right) \leq 3\sqrt{N+1}$ hold for $0 \leq i \neq j \leq 2^m$. □

For each subset, we have a LCZ with the correlation magnitude below $3\sqrt{N+1}$. This LCZ is illustrated in Fig. 3(a) where $N = 127$ and $N_{CS} = 20$. It shows the auto-correlation of the ZC_{bG} sequence $\mathbf{d}_{(1,0,8,0)}$ and its cross correlation with three other ZC_{bG} sequences. The periodic correlation between sequences from different subsets is illustrated in Fig. 3(b), where we can see that the inter-set correlation is between $2\sqrt{N+1}$ and $3\sqrt{N+1}$.

5 ZC Sequences with *p*-phase sequence covers (ZC_p)

In this section, we use the sequences in (13) to cover ZC sequences and propose a family of sequences with LCZ . We hereby define $N = p^m - 1$ until the end of this section to avoid definition confusion, where p is an odd prime number and m is a positive integer. Note that d is an integer with $1 < d < p$ in this section.

ZC_p sequences $\mathbf{g}_{(u,v,\Delta,w)}$ of period N are constructed by element-by-element multiplication of sequences $\mathbf{x}_{(u,v)}$ in (32) of period N with covers as

$$g_{(u,v,\Delta,w)}(n) = x_{(u,v)}(n) \cdot b_{(\Delta,w)}(n), \quad 0 \leq n \leq N-1, \qquad (46)$$

where the cover sequences $\mathbf{b}_{(\Delta,w)} = (b_{(\Delta,w)}(0), b_{(\Delta,w)}(1), \cdots, b_{(\Delta,w)}(N-1))$ with $0 \leq w \leq N-1$ are cyclic-shifts of the cover in (13)

$$b_{(\Delta,w)}(n) = b_\Delta(\langle n + w\rangle_N), \quad 0 \leq n \leq N-1. \qquad (47)$$

Note that when $\Delta = (0, 0, \cdots, 0) \in \mathbb{F}_{p^m}^{d-1}$, ZC_p sequences are actually the ZC_m sequence subset with root u in [8, 9].

From this definition, we generate the following set

$$\mathcal{G} = \bigcup_{\Delta\in\mathbb{F}_{p^m}^{d-1}} \mathcal{G}_\Delta, \qquad (48)$$

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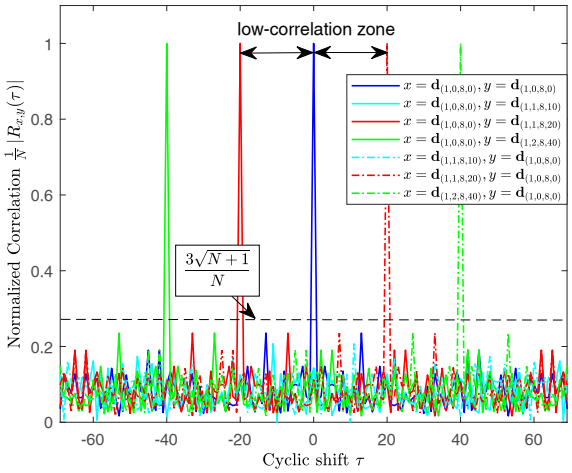
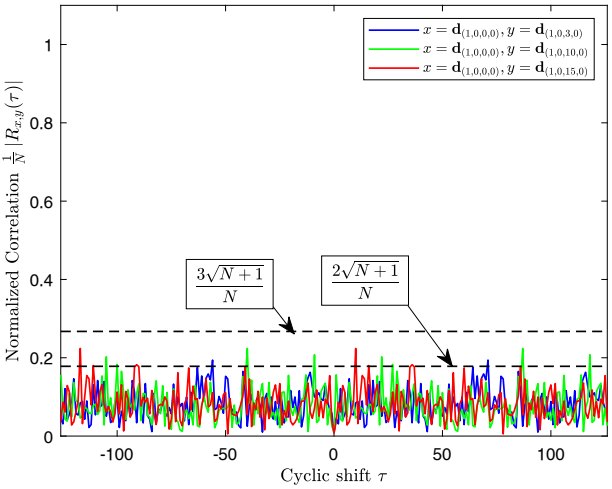


(a) Intra-subset correlation $N = 127$



(b) Inter-subset correlation $N = 127$

Fig. 3 Illustration of the LCZ of ZC_{bG} sequences

where the first subset with $\Delta = (0, 0, \cdots, 0) \in \mathbb{F}_{p^m}^{d-1}$, which contains cyclically-shifted ZC sequences with root u and ZC sequences with p-ary m-sequence covers, is defined as

$$\mathcal{G}_{\mathbf{0}} = \{\mathbf{x}_{(u,v)}\}_{v=0}^{\lfloor \frac{N}{N_{CS}} \rfloor - 1} \bigcup_{w=0}^{N-1} \{\mathbf{g}_{(u,v,\mathbf{0},w)}(n)\}_{v=0}^{\lfloor \frac{N}{N_{CS}} \rfloor - 1}, \tag{49}$$

where $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{F}_{p^m}^{d-1}$ and for $\Delta \in \mathbb{F}_{p^m}^{d-1}$ and $\Delta \neq \mathbf{0}$, subsets are defined as

$$\mathcal{G}_{\Delta} = \bigcup_{w=0}^{N-1} \{\mathbf{g}_{(u,v,\Delta,w)}(n)\}_{v=0}^{\lfloor \frac{N}{N_{CS}} \rfloor - 1}, \tag{50}$$

where u is a constant with $gcd(u, N) = 1$ for all Δ.

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From the above construction, we can see that the size of the sequence subset with index $\Delta = \mathbf{0}$ is $(N+1)\lfloor\frac{N}{N_{CS}}\rfloor$, while the size of other sequence subsets for all index $\Delta \in \mathbb{F}_{p^m}^{d-1}$ and $\Delta \neq \mathbf{0}$ is $N\lfloor\frac{N}{N_{CS}}\rfloor$. Therefore, we have

$$\text{PRACHcapacity}^{(\text{ZC}_p)} = \left(N(N+1)^{d-1}+1\right)\lfloor\frac{N}{N_{CS}}\rfloor. \tag{51}$$

At the same time, the collection of $(N+1)^{d-1}$ subsets has the following properties:

Theorem 3 *Let p be an odd prime number, N, m and N_{CS} be three positive integers with $N = p^m - 1$ and $0 < N_{CS} \leq N$. For each subset $\mathcal{G}_\Delta$ in (49) and (50) we have*

$$\Theta(\mathcal{G}_\Delta; N_{CS}) \leq d\sqrt{N+1}, \tag{52}$$

and for any two subsets $\mathcal{G}_{\Delta_1}$ and $\mathcal{G}_{\Delta_2}$, $\Delta_1 \neq \Delta_2 \in \mathbb{F}_{p^m}^{d-1}$, we have

$$\Omega\left(\mathcal{G}_{\Delta_1}, \mathcal{G}_{\Delta_2}\right) \leq d\sqrt{N+1}, \tag{53}$$

where d is an integer with $1 < d < p$.

Proof Let us consider the correlation between $\mathcal{G}_{\Delta_1}$ and $\mathcal{G}_{\Delta_2}$, for $\Delta_1 = (\delta_{(1,1)}, \delta_{(2,1)}, \cdots, \delta_{(d-1,1)})$, $\Delta_2 = (\delta_{(1,2)}, \delta_{(2,2)}, \cdots, \delta_{(d-1,2)}) \in \mathbb{F}_{p^m}^{d-1}$, we have

$$\begin{aligned} &|R_{\mathbf{g}_{(u,v_1,\Delta_1,w_1)},\mathbf{g}_{(u,v_2,\Delta_2,w_2)}}(\tau)| \\ &= \left|\sum_{n=0}^{N-1} \zeta_N^{un(C_{v_1}-C_{v_2}-\tau)} \cdot \chi\left(f(\alpha^n)\right)\right|, \end{aligned} \tag{54}$$

where $f(x) = (\alpha^{w_1} - \alpha^{w_2+\tau})x + (\delta_{(1,1)}\alpha^{2w_1} - \delta_{(1,2)}\alpha^{2w_2+2\tau})x^2 + \cdots + (\delta_{(d-1,1)}\,\alpha^{dw_1} - \delta_{(d-1,2)}\alpha^{d(w_2+\tau)})x^d$.

When $\langle C_{v_1} - C_{v_2} - \tau\rangle_N = 0$ and $\langle w_1 - w_2 - \tau\rangle_N = 0$, we know that $\zeta_N^{un(C_{v_1}-C_{v_2}-\tau)} = 1$ and $\alpha^{w_1} = \alpha^{w_2+\tau}$, (54) can be written as

$$\begin{aligned} &|R_{\mathbf{g}_{(u,v_1,\Delta_1,w_1)},\mathbf{g}_{(u,v_2,\Delta_2,w_2)}}(\tau)| \\ &= \left|\sum_{n=0}^{N-1} \chi\left(f(\alpha^n)\right)\right| \\ &= \left|-1 + \sum_{x\in\mathbb{F}_{p^m}} \chi\left(f(x)\right)\right| \\ &\begin{cases} = p^m - 1 = N, & \text{if} \quad \Delta_1 = \Delta_2, \\ \leq 1 + (d-1)\sqrt{p^m} = 1 + (d-1)\sqrt{N+1}, & \text{if} \quad \Delta_1 \neq \Delta_2 \text{ (using Fact 1)}, \end{cases} \end{aligned} \tag{55}$$

where the second case comes from the fact that $f(x) = (\delta_{(1,1)}\alpha^{2w_1} - \delta_{(1,2)}\alpha^{2w_2+2\tau})x^2 + \cdots + (\delta_{(d-1,1)}\alpha^{dw_1} - \delta_{(d-1,2)}\alpha^{d(w_2+\tau)})x^d$ with degree d_1, and $(d_1, p^m) = 1$ for all $2 \leq d_1 \leq d < p$.

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When $\langle C_{v_1} - C_{v_2} - \tau \rangle_N = 0$ and $\langle w_1 - w_2 - \tau \rangle_N \neq 0$, we know that $\zeta_N^{un(C_{v_1} - C_{v_2} - \tau)} = 1$ and $\alpha^{w_1} \neq \alpha^{w_2+\tau}$, then the left side of (54) is equal to

$$\begin{aligned} & |R_{\mathbf{g}_{(u,v_1,\Delta_1,w_1)},\mathbf{g}_{(u,v_2,\Delta_2,w_2)}}(\tau)| \\ = & \left| \sum_{n=0}^{N-1} \chi\left(f(\alpha^n)\right) \right| \\ \leq & 1 + (d-1)\sqrt{p^m} \text{ (using Fact 1)} \\ = & 1 + (d-1)\sqrt{N+1}, \end{aligned} \tag{56}$$

where the inequality comes from the fact that $f(x) = (\alpha^{w_1} - \alpha^{w_2+\tau})x + (\delta_{(1,1)}\alpha^{2w_1} - \delta_{(1,2)}\alpha^{2w_2+2\tau})x^2 + \cdots + (\delta_{(d-1,1)}\alpha^{dw_1} - \delta_{(d-1,2)}\alpha^{d(w_2+\tau)})x^d$ with degree d_2, and $(d_2, p^m) = 1$ for all $1 \leq d_2 \leq d < p$.

When $\langle C_{v_1} - C_{v_2} - \tau \rangle_N \neq 0$ and $\langle w_1 - w_2 - \tau \rangle_N = 0$, we know that $\zeta_N^{un(C_{v_1} - C_{v_2} - \tau)} \neq 1$ and $\alpha^{w_1} = \alpha^{w_2+\tau}$. Let $s = (\langle u(C_{v_1} - C_{v_2} - \tau) \rangle_N, N)$, $M = N/s$, $r = \langle u(C_{v_1} - C_{v_2} - \tau) \rangle_N / s$. Then the left side of (54) is equal to

$$\begin{aligned} & |R_{\mathbf{g}_{(u,v_1,\Delta_1,w_1)},\mathbf{g}_{(u,v_2,\Delta_2,w_2)}}(\tau)| \\ = & \left| \sum_{n=0}^{N-1} \zeta_M^{rn} \cdot \chi\left(f(\alpha^n)\right) \right| \\ = & \left| \sum_{x \in \mathbb{F}_{p^m}} \psi(x)\chi\left(f(x)\right) \right| \\ & \begin{cases} = \left| s \sum_{n=0}^{M-1} \zeta_M^{rn} \right| = 0, & \text{if} \quad \Delta_1 = \Delta_2, \\ \leq d\sqrt{p^m} = d\sqrt{N+1}, & \text{if} \quad \Delta_1 \neq \Delta_2 \text{ (using Fact 2)}, \end{cases} \end{aligned} \tag{57}$$

where the inequality comes from the fact that $g(x) = x$ with one root in $\overline{\mathbb{F}}_{p^m}$ and $M \neq 1$, at the same time, $f(x) = (\delta_{(1,1)}\alpha^{2w_1} - \delta_{(1,2)}\alpha^{2w_2+2\tau})x^2 + \cdots + (\delta_{(d-1,1)}\alpha^{dw_1} - \delta_{(d-1,2)}\alpha^{d(w_2+\tau)})x^d$ with degree $d_3 \in [2, d]$.

When $\langle C_{v_1} - C_{v_2} - \tau \rangle_N \neq 0$ and $\langle w_1 - w_2 - \tau \rangle_N \neq 0$. The left side of (54) is equal to

$$\begin{aligned} & |R_{\mathbf{g}_{(u,v_1,\Delta_1,w_1)},\mathbf{g}_{(u,v_2,\Delta_2,w_2)}}(\tau)| \\ = & \left| \sum_{n=0}^{N-1} \zeta_M^{rn} \cdot \chi\left(f(\alpha^n)\right) \right| \\ = & \left| \sum_{x \in \mathbb{F}_{p^m}} \psi(x)\chi\left(f(x)\right) \right| \\ \leq & d\sqrt{p^m} \text{ (using Fact 2)} \\ = & d\sqrt{N+1}, \end{aligned} \tag{58}$$

where the inequality comes from the fact that $g(x) = x$ with one root in $\overline{\mathbb{F}}_{p^m}$ and $M \neq 1$, at the same time, $f(x) = (\alpha^{w_1} - \alpha^{w_2+\tau})x + (\delta_{(1,1)}\alpha^{2w_1} - \delta_{(1,2)}\alpha^{2w_2+2\tau})x^2 + \cdots + (\delta_{(d-1,1)}\alpha^{dw_1} - \delta_{(d-1,2)}\alpha^{d(w_2+\tau)})x^d$ with degree $d_4 \in [1, d]$.

Proofs for other cases can be obtained similarly.

We can see from (57) and (58) that when $\Delta_1 = \Delta_2$, the maximum correlation amplitude of $\mathcal{G}_{\Delta_1}$ in the zone of length N_{CS} is less than or equal to $d\sqrt{N+1}$, so $\Theta(\mathcal{G}_\Delta; N_{CS}) \leq d\sqrt{N+1}$ holds. At the same time, it can be seen from above four cases that the maximum cross-correlation amplitude of $\mathcal{G}_{\Delta_1}$ and $\mathcal{G}_{\Delta_2}$ is also less than or equal to $d\sqrt{N+1}$, so $\Omega\left(\mathcal{G}_{\Delta_1}, \mathcal{G}_{\Delta_2}\right) \leq d\sqrt{N+1}$ is established. □

When we set $d = 2$, we can obtain $N+1$ sequence subsets with LCZ through the above construction. This LCZ is illustrated in Fig. 4(a) where $N = 5^3 - 1 = 124$ and $N_{CS} = 20$.

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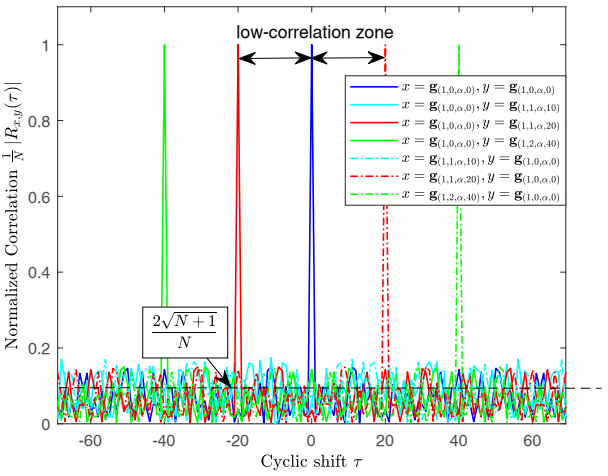
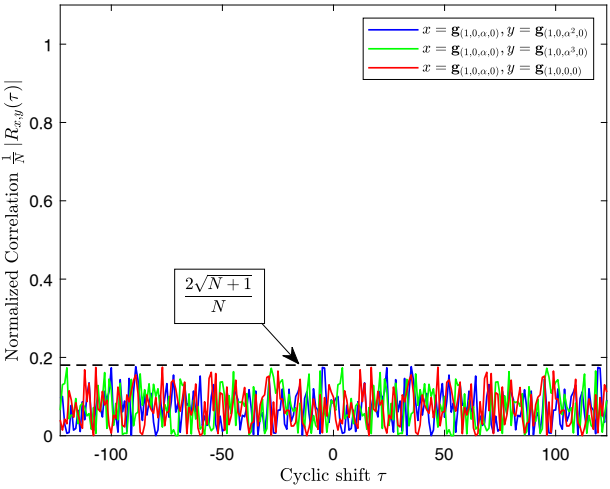


(a) Intra-subset correlation $N = 124$



(b) Inter-subset correlation $N = 124$

Fig. 4 Illustration of the LCZ of ZC_p sequences with $d = 2$

We assume that the primitive element of $\mathbb{F}_{5^3}$ is α, then Fig. 4(a) shows the auto-correlation of the ZC_p sequence $\mathbf{g}_{(1,0,\alpha,0)}$ and its cross-correlation with three other ZC_p sequences. The periodic correlation between sequences from different subsets is illustrated in Fig. 4(b), where we can see that the inter-set correlation is lower than $2\sqrt{N+1}$.

6 Comparison with previous works

In this section, we have compared the new sequences in this paper with the sequences in [8, 9]. The comparison can be summarised as follows:

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1. Considering ZC sequences and m-sequence covers in Theorem 1 have the same period, i.e. $L = N$, then the sequence family $\mathcal{S}_1$ is the ZC_m sequence subset with root u in [8, 9]. Therefore, the proposed sequence family $\mathcal{S}_1$, in Theorem 1, can be regarded as an extension of ZC_m sequence subset.
2. The sizes of sequence families in Theorem 1 and Corollary 1 can be changed with respect to L, so the proposed construction provides a flexible tradeoff between the family size and alphabet size, when the period of sequences is fixed.
3. The two ZC sequence supersets, i.e., $\mathcal{D}_{2^m}$ which is constructed using ZC sequences with binary Gold sequence covers, and $\mathcal{G}_{\mathbf{0}}$, which is constructed using ZC sequences with p-phase sequence covers, both contain the ZC_m sequence subset proposed in [8, 9].
4. For the ZC_p sequence set with $d = 2$, its inter-set correlation is $2\sqrt{N+1} \approx 2\sqrt{N}$, which is approximate to the ZC_a sequence set in [9], and our PRACH capacity is $(N^2 + N + 1)\lfloor \frac{N}{N_{CS}} \rfloor$ larger than the ZC_a set. Hence, the proposed superset is more advantageous in overcoming inter-cell interference in small cells.
5. Table 1 provides a comparison on superset optimality factors and PRACH capacities of ZC sequence supersets, proposed till date. For proposed constructions, we see that the correlation of ZC_p sets $\mathcal{G}$ is close to optimal when the parameter value $d = 2$. Also note that, the PRACH capacity of ZC_p set is much larger than $(N+1)^{d-1}$ times of the ZC sequence set as N increases.

7 Concluding and remarks

In this paper, we constructed four supersets of ZC sequences. Along with the ZC sequences, we have used m sequences, binary Gold sequences and p-phase sequences as covers. We have exploited the properties of exponential sums to find the correlation properties of the proposed supersets of ZC sequences. The parameters of the proposed supersets are more flexible and the PRACH capacities are also higher as compared to the existing ZC supersets.

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